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**REPLY TO COMMENT**

# Reply to comment on 'The Darboux transformation and algebraic deformations of shape-invariant potentials'

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**Abstract**
We reply to the comment on our recent paper made by Drs Sinha and Roy.
We agree that the backward Darboux transformation method used in our paper
is equivalent to the approach based on CES potentials, but we stress that the
emphasis and the results of our paper are different.

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We thank Sinha and Roy for pointing out the equivalence between the backward Darboux
transformation and the Junker–Roy ansatz. We were unfortunately not aware of the
contributions of Junker and Roy, and we are happy to see that proper reference to their
important work will be made in the context of our paper.

We would like to take this opportunity to point out that the emphasis and the results in
our paper are quite different from Junker and Roy. Indeed, the main theme of our paper *is not*
the backward Darboux transformation *per se*, but rather an approach to exact solvability based
on the existence of invariant infinite flags of polynomials (in a suitable coordinate system and
gauge).

The backward Darboux transformation has been applied to shape-invariant potentials
already in numerous works [5–9] to generate new exactly solvable potential forms. However,
as pointed out in [9], the transformed potential will only be defined as an integral of an
eigenfunction of the original Hamiltonian or by a formal power series. This is also the case in
[2], where the expression of the partner potential depends in all cases on the log-derivative of
$u(x)$, where $u(x)$ is a solution of the hypergeometric or confluent hypergeometric equation,
which in general will not be an elementary function. In [9] it is noted that only for certain
discrete values of the energy and shape parameter is the transformed potential an elementary
function.

In our recent paper we have investigated this subclass of *algebraic deformations*, and we have explained the interplay between these transformations and the property of exact solvability in the sense of Turbiner [10]. We show that for each shape-invariant potential there exists a countable family of algebraic deformations indexed by an integer parameter $m$. In our subsequent work [4], the structure of the infinite invariant flag associated with the $m$th deformation is further investigated. More specifically, our aims in [3] were as follows:

(i) To characterize the class of non-singular shape-invariant potentials by the property that the corresponding Schrödinger operators are precisely those which are equivalent under a change of independent variable and a gauge transformation to the operators which preserve the infinite monomial flag

$$\mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots$$
$$\mathcal{P}_n = \langle 1, z, z^2, \ldots, z^n \rangle.$$

(ii) To systematically determine the algebraic deformations of shape-invariant potentials without taking recourse to any special ansatz. We show that these correspond to a countable infinity indexed by an integer $m$.

(iii) To characterize the first deformation of the class of shape-invariant potentials by the property that the corresponding Schrödinger operators are equivalent under a change of independent variable and a gauge transformation to differential operators preserving exceptional monomial modules

$$\mathcal{P}_0^{(1)} \subset \mathcal{P}_2^{(1)} \subset \mathcal{P}_3^{(1)} \subset \cdots \subset \mathcal{P}_n^{(1)} \subset \cdots$$
$$\mathcal{P}_n^{(1)} = \langle 1, z^2, z^3, \ldots, z^n \rangle \qquad \mathcal{P}_0^{(1)} = \langle 1 \rangle.$$

(iv) To show that the algebraic deformations that have been thereby obtained do not admit an $\mathfrak{sl}(2)$ hidden symmetry algebra.

All the points written above we believe to be original. Our emphasis was not on the novelty of the potential forms, and we certainly did not claim that the deformed harmonic oscillator example is original. A detailed analysis of this example appears in a not very well known paper by Dubov *et al* [11] published in 1992, and is referenced as such in our paper.

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